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On the attainable order of collocation methods for pantograph integro-differential equations

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Abstract

For the pantograph integro-differential equation (PIDE) with nonhomogeneous term: $y'(t) = ay(t) + \int_0^1 y(\sigma(q)t)\,\mathrm{d}\mu(q) + \int_0^1 y'(\rho(q)t)\,\mathrm{d}\nu(q) + f(t)$, $t > 0$, $y(0) = y_0$, with proportional delays $\sigma(q)t$ and $\rho(q)t$, $0 < \sigma(q)$, $\rho(q) \leqslant 1$, $0 < q \leqslant 1$, we consider the attainable order of m-stage implicit (collocation-based) Runge–Kutta methods at the first mesh point $t = h$, and give conditions on the collocation polynomials $M_m(t)$ of degree m to $v(th)$, $t \in [0,1]$ such that $|v(h) - y(h)| = \mathrm{O}(h^{2m+1})$, where $y(t)$ is the solution and $v(t)$ is the collocation solution of PIDE. If $m = 2$ or $f(t)$ is a polynomial of t whose degree is less than or equal to m, then such conditions of $M_m(t)$ are simplified. A numerical example is also included.

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Keywords: Integro-differential equation; Proportional delay; Collocation method

1. Introduction

For the nonlinear delay differential equation with a constant delay, it is well known that if the collocation points are given by the Gauss–Legendre points, then the resulting implicit m-stage

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Runge–Kutta method has order $p^* = 2m$ (see [1]). For the delay Volterra integral equation with a constant delay, it is shown that the collocation solution u exhibits only order $p^* = m$, regardless of the choice of the collocation points, but if the collocation points are given by the Gauss–Legendre points, then the iterated collocation solution u_{it} has again the order $p^* = 2m$ on the mesh (see [3]).

On the other hand, for the delay differential equation (DDE): $y'(t) = by(qt)$, $y(0) = y_0$ with a proportional delay qt, $0 < q < 1$, and for the delay Volterra integral equation (DVIE): $y(t) = y_0 + \int_0^t by(qs)\,\mathrm{d}s$, Brunner [4] showed that the collocation solution $v(t)$ satisfies $v(h) = R_{2,2}(h)$, where $R_{2,2}(h)$ is the (2,2)-Padé approximant to $y(t)$ and $|v(h) - y(h)| = \mathrm{O}(h^5)$ for every $q \in (0, 1]$, if, and only if, the collocation parameters $c_j(q)$, $j = 1, 2$ are zeros of $M_2(t; q) = P_2(2s - 1) + r_1(q)P_1(2s - 1)$ with $r_1(q) = 3(1 - q)^2/(3 - 2q)$, where $P_n(t)$ is the Legendre polynomial of degree n.

As an extension of these results of Brunner [4], Takama et al. [11] showed that for $m \geqslant 1$, there exist the collocation parameters $\bar{c}_j = \bar{c}_j(q)$, $j = 1, 2, \ldots, m$ such that $0 < \bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_m < 1$, and at $t = h$, $v(t)$ is the (m, m)-Padé approximant to $y(t)$ for the DDE. Ishiwata [10] extended their results to the following neutral functional-differential equation (NFDE):

$$y'(t) = ay(t) + \sum_{i=1}^{\infty} b_i y(q_i t) + \sum_{i=1}^{\infty} c_i y'(p_i t), \quad t \geqslant 0,\ 0 < p_i, q_i < 1,\ y(0) = 1$$

with complex numbers a, b_i, c_i, and delay Volterra integro-differential equation (DVIDE):

$$y(t) = 1 + \int_0^t ay(s)\,\mathrm{d}s + \sum_{i=1}^{\infty} \int_0^t b_i y(q_i s)\,\mathrm{d}s + \sum_{i=1}^{\infty} \int_0^t c_i y'(p_i \tau)\,\mathrm{d}\tau, \quad 0 < p_i, q_i < 1.$$

In this paper, we generalize the above results to the following pantograph integro-differential equation (PIDE) with nonhomogeneous term:

$$y'(t) = ay(t) + \int_0^1 y(\sigma(q)t)\,\mathrm{d}\mu(q) + \int_0^1 y'(\rho(q)t)\,\mathrm{d}v(q) + f(t), \quad t > 0,\ y(0) = y_0, \tag{1.1}$$

where $0 < \sigma(q)$, $\rho(q) \leqslant 1$, $0 < q \leqslant 1$ and $f(t)$ is an entire complex function, a and y_0 are complex numbers, $\mu(q), v(q)$ are complex-valued functions of bounded variation on $[0, 1]$, and the integrals under considerations are Riemann–Stieltjes type. These kinds of PIDEs include many interesting applications for special cases such as collection of current by the pantograph of an electric locomotive, nonlinear dynamics, electric dynamics, probability theory on algebraic structures, partition problems in number theory, etc. (see for example, [6–8] and the references cited therein).

For m-stage implicit Runge–Kutta methods, we give conditions on the existence of the collocation solution $v(t)$ of (1.1) which satisfies $|v(h) - y(h)| = \mathrm{O}(h^{2m+1})$ at the first mesh point $t = h$. In Section 2, we give conditions on the existence of the collocation polynomials $M_m(t)$ of $v(th)$, $t \in [0, 1]$ to the solution $y(t)$, where $v(t)$ is the collocation solution of (1.1) such that $|v(h) - y(h)| = \mathrm{O}(h^{2m+1})$ (cf. [1,3]). Let $f(t)$ be a polynomial of degree l. Then we find that the conditions are classified into the following two cases: If $l \leqslant m$, then $v(h) = Q_{m,m}(h)$, which is similar to results in [10]. But if $l > m$, then $v(h) = Q_{2m,m}(h) + \mathrm{O}(h^{2m+1})$, where $Q_{m,m}(t)$ and $Q_{2m,m}(t)$ are, respectively, the (m, m) and $(2m, m)$-rational approximants to $y(t)$. If $m = 2$, such conditions of $M_m(t)$ are simplified. In Section 3, we present a numerical example.

2. Collocation methods

First, for the PIDE (1.1), assume that for all real constants s, there exist

$$\mu_s(q)=\int_0^q \sigma^s(\tau)\,\mathrm{d}\mu(\tau),\qquad v_s(q)=\int_0^q \rho^s(\tau)\,\mathrm{d}v(\tau),\qquad \mu_s^*=\int_0^1 \sigma^s(\tau)|\,\mathrm{d}\mu(\tau)|,$$

$$v_s^*=\int_0^1 \rho^s(\tau)|\,\mathrm{d}v(\tau)|\quad\text{and}\quad \lim_{h\to 0+}\int_{1-h}^1 |\,\mathrm{d}\mu(q)|=\lim_{h\to 0+}\int_{1-h}^1 |\,\mathrm{d}v(q)|=0.$$

For $f(t)\equiv 0$, (1.1) has a unique solution in $C^\infty[0,\infty)$ if, and only if, $v_n(1)\neq 1$ for all $n\in Z^+$. This solution is of the form $y(t)=y_0\sum_{n=0}^{\infty}(\prod_{k=0}^{n-1}(a+\mu_k(1))/(1-v_k(1)))t^n/n!$, and this is also given by the generalized hypergeometric function

$$y(t)=y_0\,{}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B;\mathrm{e}^{\mathrm{i}\theta}t)=y_0\sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}\frac{\prod_{i=1}^{A}(k+\alpha_i)}{\prod_{i=1}^{B}(k+\beta_i)}\right)\frac{(\mathrm{e}^{\mathrm{i}\theta}t)^n}{n!},\qquad A\leqslant B,$$

where α_i,β_i are complex constants, and $\theta\in[0,2\pi)$ (see [6–8]).

For $f(t)\not\equiv 0$, assume that the solution of (1.1) exists in $C^\infty[0,\infty)$. Then, the (unique) solution of (1.1) in $\mathrm{C}^\infty[0,\infty)$ is given by

$$y(t)=\sum_{k=0}^{\infty}\psi_k t^k,\qquad t\geqslant 0, \tag{2.1}$$

where $\psi_0=y_0$, $f(t)=\sum_{k=0}^{\infty}(f_k/k!)t^k$ and for $k=1,2,\ldots$

$$\begin{aligned}\psi_k&=\frac{(a+\int_0^1\sigma(q)^{k-1}\,\mathrm{d}\mu(q))\psi_{k-1}+f_{k-1}/(k-1)!}{k(1-\int_0^1\rho(q)^{k-1}\,\mathrm{d}v(q))}\\&=\frac{1}{k!}\left\{\left(\prod_{j=0}^{k-1}\frac{a+\mu_j(1)}{1-v_j(1)}\right)y_0+\frac{f_{k-1}}{1-v_{k-1}(1)}+\frac{a+\mu_{k-1}(1)}{1-v_{k-1}(1)}\,\frac{f_{k-2}}{1-v_{k-2}(1)}\right.\\&\quad\left.+\left(\prod_{j=k-2}^{k-1}\frac{a+\mu_j(1)}{1-v_j(1)}\right)\frac{f_{k-3}}{1-v_{k-3}(1)}+\cdots+\left(\prod_{j=1}^{k-1}\frac{a+\mu_j(1)}{1-v_j(1)}\right)\frac{f_0}{1-v_0(1)}\right\}.\end{aligned} \tag{2.2}$$

Now, using the notation of Brunner [4], we explain briefly the collocation method to obtain the collocation solution $v(t)$ of the PIDE (1.1). Assume that the PIDE (1.1) is discretized by collocation in the $(Nm+1)$-dimensional space $S_m^{(0)}(\Pi_N)$ of continuous piecewise polynomials of degree $m\geqslant 1$, with the underlying mesh $\Pi_N=\{t_n=nh : n=0,1,\ldots,N\}$, and the collocation points are given by

$$X_N=\{t_{n,j}=t_n+\bar{c}_jh : 0\leqslant\bar{c}_1<\cdots<\bar{c}_m\leqslant 1; n=0,1,\ldots,N-1\}.$$

Let v be the collocation solution satisfying (1.1) on the set X_N, and define

$$q_{n,j} = [q(n+\bar{c}_j)] \in N_0, \qquad \gamma_{n,j} = q(n+\bar{c}_j) - q_{n,j} \in [0,1), \quad j = 1,2,\ldots,m,$$

where $[x]$ denotes the greatest integer not exceeding $x \in R$.

With the above notation, we have $qt_{n,j} = h(q_{n,j} + \gamma_{n,j}) = t_{q_{n,j}} + \gamma_{n,j}h$. On $[t_n, t_{n+1}]$ the collocation equation to (1.1) is given by

$$v'(t_{n,j}) = av(t_{n,j}) + \int_0^1 v(\sigma(q)t_{n,j})\,\mathrm{d}\mu(q) + \int_0^1 v'(\rho(q)t_{n,j})\,\mathrm{d}v(q) + f(t_{n,j}), \quad j = 1,2,\ldots,m. \tag{2.3}$$

2.1. Collocation solution $v(t)$ of (1.1)

First, using the notation of Takama et al. [11] and Ishiwata [10], let us extend their results to (1.1). We consider the approximations (and their orders) at the first mesh point $t_1 = h$.

Lemma 2.1. *If for the collocation polynomial $M_m(t) = \Pi_{i=1}^m (t - \bar{c}_i)$ with distinct $\bar{c}_i$, the collocation equation (2.3) to (1.1) has a unique collocation solution $v(t)$, $0 \leqslant t \leqslant h$, then there exists an analytic function $K(t) \not\equiv 0$ such that*

$$v'(t) = av(t) + \int_0^1 v(\sigma(q)t)\,\mathrm{d}\mu(q) + \int_0^1 v'(\rho(q)t)\,\mathrm{d}v(q) + f(t) + K(t)M_m(t/h). \tag{2.4}$$

The proof of Lemma 2.1 is evident.

Note that if $f(t)$ is a polynomial of t whose degree is equal to or less than m, then $K(t) \equiv K$ (constant) in (2.4). If $f(t)$ is more general, then we need to use an analytic function $K(t) \not\equiv K$ for the generalization of (2.3).

Let us consider $\tilde{v}(t) = v(ht)$, $0 \leqslant t \leqslant 1$. Then we have $\tilde{v}(0) = y_0$ and for $0 \leqslant t \leqslant 1$,

$$\begin{aligned}\tilde{v}'(t) &= hv'(ht)\\ &= h\left\{av(ht) + \int_0^1 v(\sigma(q)ht)\,\mathrm{d}\mu(q) + \int_0^1 v'(\rho(q)ht)\,\mathrm{d}v(q) + f(ht)\right\} + K(t)M_m(t).\end{aligned}$$

Hence

$$\tilde{v}'(t) - \int_0^1 \tilde{v}'(\rho(q)t)\,\mathrm{d}v(q) = h\left\{a\tilde{v}(t) + \int_0^1 \tilde{v}(\sigma(q)t)\,\mathrm{d}\mu(q) + f(ht)\right\} + K(t)M_m(t). \tag{2.5}$$

Let the collocation polynomial $M_m(t)$ be

$$M_m(t) = \sum_{i=0}^m \frac{M_i}{i!} t^i, \quad 0 \leqslant t \leqslant 1, \; M_i = M_m^{(i)}(0), \; 0 \leqslant i \leqslant m, \; M_m \neq 0.$$

Now, by using the technique introduced by Nørsett (see [9] and its references), we obtain two lemmas that are similar to [11, Lemma 2.1; 10]. Hereafter, we denote simply $\alpha_j \equiv a + \mu_j(1)$ and $\beta_j \equiv 1 - v_j(1)$ for every j.

Lemma 2.2. *For a sufficiently small $h > 0$, assume the conditions of Lemma 2.1, and put $v(ht) = \tilde{v}(t) = \sum_{n=0}^{m}(\tilde{v}^{(n)}(0)/n!)t^n$, $0 \leqslant t \leqslant 1$ with $\tilde{v}(0) = y_0$ and $f(t) = \sum_{n=0}^{\infty}(f_n/n!)t^n$, $K(t) = \sum_{n=0}^{\infty}(K_n/n!)t^n$. Then, for $l = 1, 2, \ldots, m$,*

$$\tilde{v}^{(l)}(0) = h^l \psi_l l! + \beta_{l-1}^{-1}\left(K_0 M_{l-1} + \binom{l-1}{1} K_1 M_{l-2} + \cdots + \binom{l-1}{l-1} K_{l-1} M_0\right)$$
$$+ h\alpha_{l-1}\beta_{l-1}^{-1}\beta_{l-2}^{-1}\left(K_0 M_{l-2} + \binom{l-2}{1} K_1 M_{l-3} + \cdots + \binom{l-2}{l-2} K_{l-2} M_0\right)$$
$$+ \cdots + h^{l-2}\left(\prod_{j=2}^{l-1}\alpha_j\beta_j^{-1}\right)\beta_1^{-1}(K_0 M_1 + K_1 M_0) + h^{l-1}\left(\prod_{j=1}^{l-1}\alpha_j\beta_j^{-1}\right)\beta_0^{-1} K_0 M_0,$$

$$v(h) = \tilde{v}(1) = \tilde{v}(0) + \sum_{l=1}^{m}\frac{\tilde{v}^{(l)}(0)}{l!}$$
$$= K_0\left\{\left(\beta_{m-1}^{-1}\frac{M_{m-1}}{m!} + \beta_{m-2}^{-1}\frac{M_{m-2}}{(m-1)!} + \cdots + \beta_0^{-1}\frac{M_0}{1!}\right)\right.$$
$$+ \left(\alpha_{m-1}\beta_{m-1}^{-1}\beta_{m-2}^{-1}\frac{M_{m-2}}{m!} + \alpha_{m-2}\beta_{m-2}^{-1}\beta_{m-3}^{-1}\frac{M_{m-3}}{(m-1)!} + \cdots + \alpha_1\beta_1^{-1}\beta_0^{-1}\frac{M_0}{2!}\right)h$$
$$\left. + \cdots + \beta_{m-1}^{-1}\left(\prod_{j=1}^{m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_0}{m!}h^{m-1}\right\} + K_1\left\{\left(\beta_{m-1}^{-1}\binom{m-1}{1}\frac{M_{m-2}}{m!}\right.\right.$$
$$\left. + \beta_{m-2}^{-1}\binom{m-2}{1}\frac{M_{m-3}}{(m-1)!} + \cdots + \beta_1^{-1}\binom{1}{1}\frac{M_0}{2!}\right)$$
$$+ \left(\alpha_{m-1}\beta_{m-1}^{-1}\beta_{m-2}^{-1}\binom{m-2}{1}\frac{M_{m-3}}{m!} + \alpha_{m-2}\beta_{m-2}^{-1}\beta_{m-3}^{-1}\binom{m-3}{1}\frac{M_{m-4}}{(m-1)!}\right.$$
$$\left.\left. + \cdots + \alpha_2\beta_2^{-1}\beta_1^{-1}\binom{1}{1}\frac{M_0}{3!}\right)h + \cdots + \beta_{m-1}^{-1}\left(\prod_{j=2}^{m-1}\alpha_j\beta_{j-1}^{-1}\right)\binom{1}{1}\frac{M_0}{m!}h^{m-2}\right\}$$
$$+ \cdots + K_{m-1}\beta_{m-1}^{-1}\binom{m-1}{m-1}\frac{M_0}{m!} + \sum_{l=0}^{m}\psi_l h^l,$$

where $K_n = \mathrm{O}(h^{m+n+1}),\ n = m, m+1, \dots,$

$$h^{m+n+1} f_{m+n} + \sum_{j=n}^{m-1} \begin{pmatrix} m+n \\ j \end{pmatrix} K_j M_{m+n-j} = \mathrm{O}(h^{2m+1}), \quad n = 1, 2, \dots, m-1,$$

$$K_0 = -\frac{h^{m+1}\beta_m \psi_{m+1}(m+1)! + G}{M_m + \alpha_m \beta_{m-1}^{-1} M_{m-1} h + \cdots + (\prod_{j=1}^{m} \alpha_j \beta_{j-1}^{-1}) M_0 h^m}$$

and

$$\begin{aligned} G = {} & K_1 \left\{ \begin{pmatrix} m \\ 1 \end{pmatrix} M_{m-1} + \alpha_m \beta_{m-1}^{-1} \begin{pmatrix} m-1 \\ 1 \end{pmatrix} M_{m-2} h + \cdots + \prod_{j=1}^{m} \alpha_j \beta_{j-1}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} M_0 h^{m-1} \right\} \\ & + K_2 \left\{ \begin{pmatrix} m \\ 2 \end{pmatrix} M_{m-2} + \alpha_m \beta_{m-1}^{-1} \begin{pmatrix} m-1 \\ 2 \end{pmatrix} M_{m-3} h + \cdots + \prod_{j=2}^{m} \alpha_j \beta_{j-1}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix} M_0 h^{m-2} \right\} \\ & + \cdots + K_m \begin{pmatrix} m \\ m \end{pmatrix} M_0. \end{aligned}$$

Proof. By our assumptions, we have

$$\begin{aligned} \tilde{v}'(0) &= \beta_0^{-1}(K_0 M_0 + h\alpha_0 \tilde{v}(0) + h f_0), \\ \tilde{v}''(0) &= \beta_1^{-1}(K_0 M_1 + K_1 M_0 + h\alpha_1 \tilde{v}'(0) + h^2 f_1), \\ &\ \vdots \\ \tilde{v}^{(m)}(0) &= \beta_{m-1}^{-1}\left(K_0 M_{m-1} + \begin{pmatrix} m-1 \\ 1 \end{pmatrix} K_1 M_{m-2} + \cdots + \begin{pmatrix} m-1 \\ m-1 \end{pmatrix} K_{m-1} M_0 \right. \\ &\quad \left. + h\alpha_{m-1} \tilde{v}^{(m-1)}(0) + h^m f_{m-1}\right), \\ 0 &= \beta_m^{-1}\left(K_0 M_m + \begin{pmatrix} m \\ 1 \end{pmatrix} K_1 M_{m-1} + \cdots + \begin{pmatrix} m \\ m \end{pmatrix} K_m M_0 + h\alpha_m \tilde{v}^{(m)}(0) + h^{m+1} f_m \right), \\ &\ \vdots \\ 0 &= \beta_{m+n}^{-1}\left(\sum_{j=0}^{m} \begin{pmatrix} m+n \\ j \end{pmatrix} K_{m+n-j} M_j + h^{m+n+1} f_{m+n}\right), \quad n = 1, 2, \dots. \end{aligned}$$

For the $(m+1)$-st equation, we successively replace the above $\tilde{v}^{(j)}(0)$ of the jth equation by corresponding expressions of $M_{j-1}, M_{j-2}, \ldots, M_0$, and $\tilde{v}^{(j-1)}(0)$, $j = m, m-1, \ldots, 1$. Then, we obtain

$$\begin{aligned}
0 &= \beta_m^{-1}\left[\left(K_0M_m + \binom{m}{1}K_1M_{m-1} + \cdots + \binom{m}{m}K_mM_0\right) + h^{m+1}f_m\right.\\
&\quad + h\alpha_m\beta_{m-1}^{-1}\left\{\left(K_0M_{m-1} + \binom{m-1}{1}K_1M_{m-2} + \cdots + \binom{m-1}{m-1}K_{m-1}M_0\right)\right.\\
&\quad \left.\left. + h\alpha_{m-1}\tilde{v}^{(m-1)}(0) + h^m f_{m-1}\right\}\right]\\
&= \beta_m^{-1}\left\{\left(K_0M_m + \binom{m}{1}K_1M_{m-1} + \cdots + \binom{m}{m}K_mM_0\right)\right.\\
&\quad \left. + h\alpha_m\beta_{m-1}^{-1}\left(K_0M_{m-1} + \binom{m-1}{1}K_1M_{m-2} + \cdots + \binom{m-1}{m-1}K_{m-1}M_0\right)\right\}\\
&\quad + h^2\alpha_m\alpha_{m-1}\beta_m^{-1}\beta_{m-1}^{-1}\tilde{v}^{(m-1)}(0) + h^{m+1}(\beta_m^{-1}f_m + \alpha_m\beta_m^{-1}\beta_{m-1}^{-1}f_{m-1})\\
&=\\
&\vdots\\
&= \beta_m^{-1}\left\{\left(K_0M_m + \binom{m}{1}K_1M_{m-1} + \cdots + \binom{m}{m}K_mM_0\right)\right.\\
&\quad + h\alpha_m\beta_{m-1}^{-1}\left(K_0M_{m-1} + \binom{m-1}{1}K_1M_{m-2} + \cdots + \binom{m-1}{m-1}K_{m-1}M_0\right)\\
&\quad + h^2\alpha_m\alpha_{m-1}\beta_{m-1}^{-1}\beta_{m-2}^{-1}\left(K_0M_{m-2}\binom{m-2}{1}K_1M_{m-2} + \cdots + \binom{m-2}{m-2}K_{m-2}M_0\right)\\
&\quad \left. + \cdots + h^m\left(\prod_{j=1}^{m}\alpha_j\beta_{j-1}^{-1}\right)K_0M_0\right\} + h^{m+1}\left\{\left(\prod_{j=0}^{m}\alpha_j\beta_j^{-1}\right)\tilde{v}(0) + \beta_m^{-1}f_m\right.\\
&\quad \left. + \alpha_m\beta_m^{-1}\beta_{m-1}^{-1}f_{m-1} + \cdots + \left(\prod_{j=2}^{m}\alpha_j\beta_j^{-1}\right)\beta_1^{-1}f_1 + \left(\prod_{j=1}^{m}\alpha_j\beta_j^{-1}\right)\beta_0^{-1}f_0\right\}.
\end{aligned}$$

Hence for a sufficiently small $h > 0$ and $M_m \neq 0$, we obtain from the above $(m+1)$-st equation,

$$K_0 = -\frac{h^{m+1}\beta_m\psi_{m+1}(m+1)! + G}{M_m + \alpha_m\beta_{m-1}^{-1}M_{m-1}h + \cdots + (\prod_{j=1}^{m}\alpha_j\beta_{j-1}^{-1})M_0h^m}.$$

Similarly, from the above lth equation, we get successively $\tilde{v}^{(l)}(0)$, $l=m,m-1,\ldots,1$, and $v(h)$ in Lemma 2.2. □

By Lemma 2.2, we easily obtain the following lemma.

Lemma 2.3. *In Lemma 2.2, assume that the $(m-1)\times(m-1)$ upper triangular matrix $A=[a_{i,j}]$ given by*

$$a_{i,j}=\begin{cases}\begin{pmatrix} m+i \\ j\end{pmatrix} M_{m+i-j}, & i\leqslant j,\\ 0, & i>j\end{cases}$$

is nonsingular. Then for its inverse matrix $A^{-1}=[a_{i,j}^{(-1)}]$, it holds that

$$a_{i,j}^{(-1)}=0,\quad 1\leqslant j\leqslant i-1,\quad a_{i,i}^{(-1)}=\frac{1}{a_{i,i}}=\frac{1}{\begin{pmatrix} m+i \\ i\end{pmatrix} M_m},\quad i=1,2,\ldots,m$$

and

$$\begin{cases} K_i=-\displaystyle\sum_{j=i}^{m-1} a_{i,j}^{(-1)} f_{m+j}h^{m+j+1}+\mathrm{O}(h^{2m+1}), & i=1,2,\ldots,m-1\\ K_j=\mathrm{O}(h^{m+j+1}), & j=m,m+1,\ldots. \end{cases}$$

In particular, if $f(t)$ is a polynomial of t whose degree is equal to or less than m, then $K_j=0$, $j=1,2,\ldots$ and for $k=m+1,m+2,\ldots,$ we have

$$\psi_k=\alpha_{k-1}\beta_{k-1}^{-1}\frac{\psi_{k-1}}{k}=\frac{\psi_m}{k(k-1)\cdots(m+1)}\prod_{j=m}^{k-1}\alpha_j\beta_j^{-1}.$$

We obtain the following theorem (see [10]).

Theorem 2.1. *Assume the conditions in Lemmas 2.2 and 2.3. Then the collocation solution $v(t)$ corresponding to the solution $y(t)$ of (1.1) has the property that*

$$v(h)=\frac{\Gamma_0+\Gamma_1 h+\Gamma_2 h^2+\cdots+\Gamma_{2m}h^{2m}}{\Lambda_0+\Lambda_1 h+\Lambda_2 h^2+\cdots+\Lambda_m h^m}+\mathrm{O}(h^{2m+1}),$$

$$\begin{cases} \Lambda_0=M_m,\quad \Lambda_l=\left(\displaystyle\prod_{j=m-l+1}^{m}\alpha_j\beta_{j-1}^{-1}\right)M_{m-l}, & l=1,2,\ldots,m,\\ \Gamma_l=\displaystyle\sum_{k=0}^{l}\psi_{l-k}\Lambda_k, & l=0,1,2,\ldots,m \end{cases}$$

and

$$\begin{cases} \Gamma_{m+1} = \displaystyle\sum_{l=1}^{m} \psi_{m+1-l}\Lambda_l + \bar{d}_{0,m+1}, \\ \Gamma_{m+i} = \displaystyle\sum_{l=i}^{m} \psi_{m+i-l}\Lambda_l + \bar{d}_{0,m+i} + \bar{e}_{1,m+i} + \cdots + \bar{e}_{i-1,m+i}, \quad i = 2,3,\ldots,m, \end{cases} \tag{2.6}$$

where

$$\begin{cases} v(h) = K_0(c_{0,0} + c_{0,1}h + \cdots + c_{0,m-1}h^{m-1}) + K_1(c_{1,0} + c_{1,1}h + \cdots + c_{1,m-2}h^{m-2}) \\ \qquad + \cdots + K_{m-1}c_{m-1,0} + \displaystyle\sum_{l=0}^{m} \psi_l h^l, \\ K_0 = \{d_{0,m+1}h^{m+1} + K_1(d_{1,0} + d_{1,1}h + \cdots + d_{1,m-1}h^{m-1}) + K_2(d_{2,0} + d_{2,1}h \\ \qquad + \cdots + d_{2,m-2}h^{m-2}) + \cdots + K_m d_{m,0}\}/(\Lambda_0 + \Lambda_1 h + \cdots + \Lambda_m h^m), \end{cases} \tag{2.7}$$

$$\begin{cases} c_{i,j} = \displaystyle\sum_{k=0}^{m-1-i-j} \beta_{i+k+j}^{-1} \left(\prod_{l=i+k+1}^{i+k+j} \alpha_l \beta_{l-1}^{-1}\right) \binom{i+k}{i} \frac{M_k}{(i+j+k+1)!}, \\ \quad 0 \leqslant i \leqslant m-1, \quad 0 \leqslant j \leqslant m-1-i, \\ d_{0,m+1} = -\beta_m \psi_{m+1}(m+1)!, \\ d_{i,j} = -\left(\displaystyle\prod_{l=m-j+1}^{m} \alpha_l \beta_{l-1}^{-1}\right) \binom{m-j}{i} M_{m-i-j}, \quad 1 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant m-i, \\ e_{i,j} = -a_{i,j-(m+1)}^{(-1)} f_{j-1}, \quad m+i+1 \leqslant j \leqslant 2m \end{cases} \tag{2.8}$$

and

$$\begin{cases} \bar{d}_{0,m+j} = d_{0,m+1}c_{0,j-1}, \quad 1 \leqslant j \leqslant m, \\ \bar{d}_{i,j} = (d_{i,0}c_{0,j} + d_{i,1}c_{0,j-1} + \cdots + d_{i,j}c_{0,0}) + (c_{i,0}\Lambda_j + c_{i,1}\Lambda_{j-1} + \cdots + c_{i,j}\Lambda_0), \\ \quad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant j \leqslant m-i-1, \\ \bar{e}_{i,j} = e_{i,j}\bar{d}_{i,0} + e_{i,j-1}\bar{d}_{i,1} + \cdots + e_{i,m+i+1}\bar{d}_{i,j-(m+i+1)}, \quad 1 \leqslant i \leqslant m-1, \quad m+i+1 \leqslant j \leqslant 2m. \end{cases} \tag{2.9}$$

Hence, we have

$$
\left\{
\begin{aligned}
\Gamma_{m+1} &= \sum_{l=0}^{m} \psi_{m+1-l}\Lambda_l - \beta_m\psi_{m+1}(m+1)!\sum_{l=0}^{m}\beta_l^{-1}\frac{M_l}{(l+1)!},\\
\Gamma_{m+2} &= \sum_{l=0}^{m} \psi_{m+2-l}\Lambda_l - \beta_m\psi_{m+1}(m+1)!\sum_{l=0}^{m}\alpha_{l+1}\beta_{l+1}^{-1}\beta_l^{-1}\frac{M_l}{(l+2)!}\\
&\quad - f_{m+1}\left\{\beta_{m+1}^{-1}\frac{M_m}{(m+2)!} + \frac{1}{\binom{m+1}{1}M_m}\left(-\binom{m}{1}M_{m-1}\right.\right.\\
&\quad\left.\left.\times\sum_{l=0}^{m-1}\beta_l^{-1}\binom{l}{0}\frac{M_l}{(l+1)!} + M_m\sum_{l=0}^{m-2}\beta_{l+1}\binom{l+1}{1}\frac{M_l}{(l+2)!}\right)\right\},\\
\Gamma_{m+3} &= \sum_{l=0}^{m} \psi_{m+3-l}\Lambda_l - \beta_m\psi_{m+1}(m+1)!\sum_{l=0}^{m}\alpha_{l+2}\alpha_{l+1}\beta_{l+2}^{-1}\beta_{l+1}^{-1}\beta_l^{-1}\frac{M_l}{(l+3)!}\\
&\quad - f_{m+1}\left(\alpha_{m+2}\beta_{m+2}^{-1}\beta_{m+1}^{-1}\frac{M_m}{(m+3)!} + \alpha_{m+1}\beta_{m+1}^{-1}\beta_m^{-1}\frac{M_{m-1}}{(m+2)!} + a_{1,1}^{(-1)}\bar{d}_{1,1}\right)\\
&\quad - f_{m+2}\left\{\beta_{m+2}^{-1}\frac{M_m}{(m+3)!} + (a_{1,2}^{(-1)}\bar{d}_{1,0} + a_{2,2}^{(-1)}\bar{d}_{2,0})\right\},\\
&\ \ \vdots\\
\Gamma_{2m} &= \sum_{l=0}^{m} \psi_{2m-l}\Lambda_l - \beta_m\psi_{m+1}(m+1)!\sum_{l=0}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+1}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!}\\
&\quad - f_{m+1}\left\{\sum_{l=2}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+2}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!} + a_{1,1}^{(-1)}\bar{d}_{1,m-2}\right\}\\
&\quad - f_{m+2}\left\{\sum_{l=3}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+3}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!} + (a_{1,2}^{(-1)}\bar{d}_{1,m-3} + a_{2,2}^{(-1)}\bar{d}_{2,m-3})\right\}\\
&\quad - \cdots - f_{2m-1}\left\{\beta_{2m-1}^{-1}\frac{M_m}{(2m)!} + (a_{1,m-1}^{(-1)}\bar{d}_{1,0} + a_{2,m-1}^{(-1)}\bar{d}_{2,0}\right.\\
&\quad\left. + \cdots + a_{m-1,m-1}^{(-1)}\bar{d}_{m-1,0})\right\}.
\end{aligned}
\right.
\tag{2.10}
$$

Proof. By Lemma 2.2, we see that (2.7) and (2.8) hold. Hence, we obtain $\Gamma_0 = \Lambda_0 = \psi_0\Lambda_0$ and for $l = 1, 2, \dots, m$, $\Gamma_l = \psi_l\Lambda_0 + \psi_{l-1}\Lambda_1 + \cdots + \psi_0\Lambda_l$. From (2.7) and (2.8) we are led, after some lengthy but straightforward calculations to (2.9) and (2.10). □

Remark 2.1. Theorem 2.1 indicates that the collocation solution $v(t)$ of (1.1) is easily determined once the collocation polynomial $M_m(t)$ has been found. However, the conditions that the zeros $\{\bar{c}_j\}_{j=1}^m$ of $M_m(t)$ are real and $0 \leqslant \bar{c}_1 < \bar{c}_2 < \cdots < \bar{c}_m \leqslant 1$, are not always satisfied. So, our collocation method (2.4) is an extended one of collocation method (2.3) to the Eq. (1.1).

2.2. Conditions on attainable order of collocation methods

For the power series $y(t) = \psi_0 + \psi_1 t + \psi_2 t^2 + \cdots$, the rational function

$$\frac{\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_m t^m}{\lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n}$$

is called an (m,n)-rational approximant to $y(t)$. We refer to this as $Q_{m,n}(t)$. In particular, if $Q_{m,n}(t)$ satisfies the following condition:

$$\begin{aligned}&(\lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n)(\psi_0 + \psi_1 t + \psi_2 t^2 + \cdots) - (\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_m t^m)\\ &\quad = \xi_{m+n+1} t^{m+n+1} + \xi_{m+n+2} t^{m+n+2} + \cdots,\end{aligned}$$

then this rational function is called the (m,n)-Padé approximant to $y(t)$. We refer to this as $R_{m,n}(t)$.

We first state a classical result of approximation theory (see [9] and its references).

Lemma 2.4 (cf. Takama et al. [11, Lemma 2.2]). *A $(2m,m)$-rational approximant to the above $y(t)$,*

$$Q_{2m,m}(t) = \frac{\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_{2m} t^{2m}}{\lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_m t^m}$$

exists and $|Q_{2m,m}(t) - y(t)| = \mathrm{O}(t^{2m+1})$ if, and only if, for $\lambda_0 \neq 0$, the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ satisfy the following linear system:

$$\begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_m \\ \psi_2 & \psi_3 & \psi_4 & \cdots & \psi_{m+1} \\ \psi_3 & \psi_4 & \psi_5 & \cdots & \psi_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_m & \psi_{m+1} & \psi_{m+2} & \cdots & \psi_{2m-1} \end{pmatrix} \begin{pmatrix} \lambda_m \\ \lambda_{m-1} \\ \lambda_{m-2} \\ \vdots \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \gamma m + 1 - \psi_{m+1}\lambda_0 \\ \gamma m + 2 - \psi_{m+2}\lambda_0 \\ \gamma m + 3 - \psi_{m+3}\lambda_0 \\ \vdots \\ \gamma_{2m} - \psi_{2m}\lambda_0 \end{pmatrix} \tag{2.11}$$

and $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m$ are determined by

$$\begin{pmatrix} \psi_0 & & & & \mathbf{0} \\ \psi_1 & \psi_0 & & & \\ \psi_2 & \psi_1 & \psi_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \psi_m & \psi_{m-1} & \psi_{m-2} & \cdots & \psi_0 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{pmatrix}. \tag{2.12}$$

In particular, if $\gamma_{m+n}=0,\ n=1,2,\ldots,m,$ *then* $Q_{2m,m}(t)=R_{m,m}(t)$ *is the* (m,m)*-Padé approximant to* $y(t)$.

Note that there is a case where the determinant of the matrix in (2.11) is zero, but there are solutions $\lambda_1,\lambda_2,\ldots,\lambda_m$ of (2.11) (see [11]).

For the existence of the collocation solution $v(t)$ to (1.1) which satisfies the condition $|v(h)-y(h)|=\mathrm{O}(h^{2m+1})$, we offer the following theorem.

Theorem 2.2. *In Theorem* 2.1, *if*

$$\Gamma_{m+i}=\sum_{l=0}^{m}\psi_{m+i-l}\Lambda_l,\quad 1\leqslant i\leqslant m, \tag{2.13}$$

that is, there are solutions $\{M_k\}_{k=0}^{m}$ *such that*

$$\begin{cases}
\beta_m\psi_{m+1}(m+1)!\displaystyle\sum_{l=0}^{m}\beta_l^{-1}\frac{M_l}{(l+1)!}=0,\\
\beta_m\psi_{m+1}(m+1)!\displaystyle\sum_{l=0}^{m}\alpha_{l+1}\beta_{l+1}^{-1}\beta_l^{-1}\frac{M_l}{(l+2)!}+\frac{f_{m+1}}{m+1}\sum_{l=0}^{m}\beta_{l+1}^{-1}\binom{l+1}{1}\frac{M_l}{(l+2)!}=0,\\
\beta_m\psi_{m+1}(m+1)!\displaystyle\sum_{l=0}^{m}\alpha_{l+2}\alpha_{l+1}\beta_{l+2}^{-1}\beta_{l+1}^{-1}\beta_l^{-1}\frac{M_l}{(l+3)!}\\
\quad+f_{m+1}\left(\alpha_{m+2}\beta_{m+2}^{-1}\beta_{m+1}^{-1}\dfrac{M_m}{(m+3)!}+\alpha_{m+1}\beta_{m+1}^{-1}\beta_m^{-1}\dfrac{M_{m-1}}{(m+2)!}+a_{1,1}^{(-1)}\bar d_{1,1}\right)\\
\quad+f_{m+2}\left\{\beta_{m+2}^{-1}\dfrac{M_m}{(m+3)!}+(a_{1,2}^{(-1)}\bar d_{1,0}+a_{2,2}^{(-1)}\bar d_{2,0})\right\}=0,\\
\quad\vdots\\
\beta_m\psi_{m+1}(m+1)!\displaystyle\sum_{l=0}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+1}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!}\\
\quad+f_{m+1}\left\{\displaystyle\sum_{l=2}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+2}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!}+a_{1,1}^{(-1)}\bar d_{1,m-2}\right\}\\
\quad+f_{m+2}\left\{\displaystyle\sum_{l=3}^{m}\beta_{l+m-1}^{-1}\left(\prod_{j=l+3}^{l+m-1}\alpha_j\beta_{j-1}^{-1}\right)\frac{M_l}{(l+m)!}+(a_{1,2}^{(-1)}\bar d_{1,m-3}+a_{2,2}^{(-1)}\bar d_{2,m-3})\right\}\\
\quad+\cdots+f_{2m-1}\left\{\beta_{2m-1}^{-1}\dfrac{M_m}{(2m)!}+(a_{1,m-1}^{(-1)}\bar d_{1,0}+a_{2,m-1}^{(-1)}\bar d_{2,0}+\cdots+a_{m-1,m-1}^{(-1)}\bar d_{m-1,0})\right\}=0,
\end{cases} \tag{2.14}$$

then $v(h)=\bar{Q}_{2m,m}(h)+\mathrm{O}(h^{2m+1})$ *and* $|v(h)-y(h)|=\mathrm{O}(h^{2m+1})$, *where* $\bar{Q}_{2m,m}(h)=(\Gamma_0+\Gamma_1 h+\Gamma_2 h^2+\cdots+\Gamma_{2m}h^{2m})/(\Lambda_0+\Lambda_1 h+\Lambda_2 h^2+\cdots+\Lambda_m h^m)$ *is determined by Theorem* 2.1 *for the collocation polynomial*

$$M_m(t)\equiv\frac{M_m}{m!}t^m+\frac{M_{m-1}}{(m-1)!}t^{m-1}+\cdots+\frac{M_1}{1!}t+M_0.$$

Proof. By (2.13), we see that $\lambda_l=\Lambda_l,\ l=0,1,2,\ldots,m$ and $\gamma_{m+l}=\Gamma_{m+l},\ l=1,2,\ldots,m$ satisfy (2.11). Hence by Lemma 2.4 and Theorem 2.1, we have $v(h)=\bar{Q}_{2m,m}(h)+\mathrm{O}(h^{2m+1})$ and $|v(h)-y(h)|=\mathrm{O}(h^{2m+1})$. (2.13) is equivalent to

$$\begin{cases}\psi_{m+1}\Lambda_0=\bar{d}_{0,m+1},\\ \displaystyle\sum_{l=0}^{i-1}\psi_{m+i-l}\Lambda_l=\bar{d}_{0,m+i}+\bar{e}_{1,m+i}+\bar{e}_{2,m+i}+\cdots+\bar{e}_{i-1,m+i},\qquad i=2,3,\ldots,m.\end{cases}$$

By Lemma 2.3, (2.8) and (2.9), we have $a_{1,1}^{(-1)}\bar{d}_{1,0}=1/\binom{m+1}{1}\sum_{l=0}^{m-1}\binom{m+1}{1}M_l/(l+1)!$. Using (2.2), (2.10) and the above relations, we get (2.14). □

Remark 2.2. For the cases that there are j and l such that $0\leqslant j\leqslant m$ and $\mu_j(1)\neq 0$ or $v_j(1)\neq 0$, and $1\leqslant l\leqslant m-1$ and $f_{m+l}\neq 0$, then $\{M_k\}_{k=0}^m$ which satisfy condition (2.14), depend on these $\{f_{m+l}\}_{l=1}^{m-1}$. For $m=2$, we easily solve (2.14) (see (2.15)–(2.17) in Example 2.1). For $m\geqslant 3$ and $\sigma(q)$ or $\rho(q)\not\equiv 1,\ 0\leqslant q\leqslant 1$, condition (2.14) becomes in general, a nonlinear problem, and to get the collocation polynomial which satisfies condition (2.14), we need some iterative methods (see (2.18) for $m=3$).

If (2.14) has solutions $\{M_k\}_{k=0}^m$, then $\bar{Q}_{2m,m}(h)$ is easily obtained by Theorem 2.1. Note that there are cases where the solutions $\{M_k\}_{k=0}^m$ of (2.14) do not exist (see Example 2.3 in [11]).

For the case of $\sigma(q)=\rho(q)\equiv 1,\ 0\leqslant q\leqslant 1$, (1.1) becomes the following differential equation with no delays:

$$y'(t)=\bar{a}y(t)+\bar{f}(t),\quad t>0\quad\text{and}\quad y(0)=y_0,$$

where $\bar{a}=(a+\int_0^1 \mathrm{d}\mu(q))/(1-\int_0^1 \mathrm{d}v(q))$ and $\bar{f}(t)=f(t)/(1-\int_0^1 \mathrm{d}v(q))=\sum_{n=0}^{\infty}(\bar{f}_n/n!)t^n$. Then we have the following well-known results (see [9] and cf. [1]).

Corollary 2.1. *If* $\sigma(q)=\rho(q)\equiv 1,\ 0\leqslant q\leqslant 1$ *and* $\{M_k\}_{k=0}^m$ *correspond to Gauss–Legendre polynomial of degree* m, *that is*,

$$M_m(t)=\frac{m!}{(2m)!}P_m(2t-1)=\frac{M_m}{m!}t^m+\frac{M_{m-1}}{(m-1)!}t^{m-1}+\cdots+M_0,$$

then $\sum_{l=0}^m M_l/(l+i)!=0,\ 1\leqslant i\leqslant m$, *and* (2.14) *is satisfied. Then the obtained* $\bar{Q}_{2m,m}(t)$ *in Theorem* 2.1 *satisfies* $|\bar{Q}_{2m,m}(h)-y(h)|=\mathrm{O}(h^{2m+1})$, *although in general, it is not the* (m,m)*-Padé approximation to* $y(t)$.

Note that for the collocation solution $v(t)$, $v(h)=\bar{Q}_{2m,m}(h)+\mathrm{O}(h^{2m+1})$.

Example 2.1. (i) For $m=1$, condition (2.14) is expressed as

$$\beta_1\psi_2 2!\left(\beta_1^{-1}\frac{M_1}{2!}+\beta_0^{-1}\frac{M_0}{1!}\right)=0$$

from which we have $\psi_2=0$ or $M_0/M_1=-\frac{1}{2}\beta_0\beta_1^{-1}$. In particular, if $\psi_2\neq 0$ and $\rho(q)\equiv 1,\ 0\leqslant q\leqslant 1$, then we have $M_0/M_1=-\frac{1}{2},\ M_1(t)=(M_1/1!)t+M_0=(M_1/2)P_1(2t-1)$,

$$\bar{Q}_{2,1}(t)=\frac{y_0+((\bar{a}/2)y_0+\bar{f}_0)t+(\bar{f}_1/2)t^2}{1-(\bar{a}/2)t} \quad\text{and}\quad |\bar{Q}_{2,1}(h)-y(h)|=\mathrm{O}(h^3).$$

(ii) For $m=2$, condition (2.14) is expressed as follows:

$$\begin{cases} \beta_2^{-1}\dfrac{M_2}{3!}+\beta_1^{-1}\dfrac{M_1}{2!}+\beta_0^{-1}\dfrac{M_0}{1!}=0, \\ \beta_2\psi_3 3!\left(\alpha_3\beta_3^{-1}\beta_2^{-1}\dfrac{M_2}{4!}+\alpha_2\beta_2^{-1}\beta_1^{-1}\dfrac{M_1}{3!}+\alpha_1\beta_1^{-1}\beta_0^{-1}\dfrac{M_0}{2!}\right) \\ \quad +f_3\left\{\beta_3^{-1}\dfrac{M_2}{4!}+\dfrac{1}{3}\left(2\beta_1^{-1}\dfrac{M_1}{3!}+\beta_1^{-1}\dfrac{M_0}{2!}\right)\right\}=0. \end{cases} \tag{2.15}$$

Hence we have that if $A_{1,1}A_{2,2}-A_{2,1}A_{1,2}\neq 0$, then

$$\frac{M_1}{M_2}=\frac{A_{2,2}B_1-A_{1,2}B_2}{A_{1,1}A_{2,2}-A_{2,1}A_{1,2}} \quad\text{and}\quad \frac{M_0}{M_2}=\frac{-A_{2,1}B_1+A_{1,1}B_2}{A_{1,1}A_{2,2}-A_{2,1}A_{1,2}}, \tag{2.16}$$

where

$$\begin{cases} A_{1,1}=\beta_1^{-1}\dfrac{1}{2!},\quad A_{1,2}=\beta_0^{-1},\quad A_{2,1}=\alpha_2\beta_1^{-1}\psi_3+\dfrac{2}{3}\beta_2^{-1}\dfrac{f_3}{3!}, \\ A_{2,2}=3\beta_2\alpha_1\beta_1^{-1}\beta_0^{-1}\psi_3+\beta_1^{-1}\dfrac{f_3}{3!}, \\ B_1=-\beta_2^{-1}\dfrac{1}{3!},\quad B_2=-\alpha_3\beta_3^{-1}\dfrac{\psi_3}{4}-\beta_3^{-1}\dfrac{f_3}{4!}. \end{cases} \tag{2.17}$$

In particular, if $\sigma(q)=\rho(q)\equiv 1,\ 0\leqslant q\leqslant 1$, then we have $|\bar{Q}_{4,2}(h)-y(h)|=\mathrm{O}(h^5)$, where $M_1/M_2=-\frac{1}{2},\ M_0/M_2=\frac{1}{12},\ M_2(t)=(M_2/2!)t^2+(M_1/1!)t+M_0=(M_2/12)P_2(2t-1)$, and

$$\bar{Q}_{4,2}(t) = \frac{y_0+((\bar{a}/2)y_0+\bar{f}_0)t+((\bar{a}^2/12)y_0+\bar{f}_1/2)t^2-((\bar{a}\bar{f}_1/12)-\bar{f}_2/6)t^3-((\bar{a}\bar{f}_2-\bar{f}_3)/24)t^4}{1-(\bar{a}/2)t+(\bar{a}^2/12)t^2}.$$

(iii) For $m=3$, condition (2.14) is expressed as follows:

$$\begin{cases} \beta_3^{-1}\dfrac{M_3}{4!}+\beta_2^{-1}\dfrac{M_2}{3!}+\beta_1^{-1}\dfrac{M_1}{2!}+\beta_0^{-1}\dfrac{M_0}{1!}=0, \\ \beta_3\psi_4 4!\alpha_4\beta_4^{-1}\beta_3^{-1}\dfrac{M_3}{5!}+\alpha_3\beta_3^{-1}\beta_2^{-1}\dfrac{M_2}{4!}+\alpha_2\beta_2^{-1}\beta_1^{-1}\dfrac{M_1}{3!}+\alpha_1\beta_1^{-1}\beta_0^{-1}\dfrac{M_0}{2!} \\ \quad +\dfrac{f_4}{4}\left(4\beta_4^{-1}\dfrac{M_3}{5!}+3\beta_3^{-1}\dfrac{M_2}{4!}+2\beta_2^{-1}\dfrac{M_1}{3!}+\beta_1^{-1}\dfrac{M_0}{2!}\right)=0, \\ \beta_3\psi_4 4!\displaystyle\sum_{l=0}^{3}\alpha_{l+2}\alpha_{l+1}\beta_{l+2}^{-1}\beta_{l+1}^{-1}\beta_l^{-1}\dfrac{M_l}{(l+3)!} \\ \quad +f_4\left(\alpha_5\beta_5^{-1}\beta_4^{-1}\dfrac{M_3}{6!}+\alpha_4\beta_4^{-1}\beta_3^{-1}\dfrac{M_2}{5!}+C_1\right)+f_5\left\{\beta_5^{-1}\dfrac{M_3}{6!}+(C_2+C_3)\right\}=0, \end{cases} \tag{2.18}$$

where

$$\begin{cases} C_1=\dfrac{1}{4M_3}\bar{d}_{1,1}=\dfrac{M_2}{4M_3}\left\{-3\left(\alpha_2\beta_2^{-1}\beta_1^{-1}\dfrac{M_1}{3!}+\alpha_1\beta_1^{-1}\beta_0^{-1}\dfrac{M_0}{2!}\right)\right. \\ \quad \left.+\alpha_2\beta_2^{-1}\left(2\beta_2^{-1}\dfrac{M_1}{3!}+\beta_1^{-1}\dfrac{M_0}{2!}\right)\right\}+2\alpha_3\beta_3^{-1}\beta_2^{-1}\dfrac{M_1}{4!}+\alpha_2\beta_2^{-1}\beta_1^{-1}\dfrac{M_0}{3!}, \\ C_2=\dfrac{-3M_2}{20M_3^2}\bar{d}_{1,0}=M_2\left\{\dfrac{-3}{20M_3}\left(3\beta_3^{-1}\dfrac{M_2}{4!}+2\beta_2^{-1}\dfrac{M_0}{2!}\right)\right\}, \\ C_3=\dfrac{1}{10M_3}\bar{d}_{2,0}=\dfrac{1}{10}\left(3\beta_3^{-1}\dfrac{M_1}{4!}+\beta_2^{-1}\dfrac{M_0}{3!}\right). \end{cases} \tag{2.19}$$

Note Remark 2.2 and

$$a_{i,i}^{(-1)}\bar{d}_{i,0}=\frac{1}{\begin{pmatrix} m+i \\ i\end{pmatrix}}\left\{\begin{pmatrix} m \\ i\end{pmatrix}\beta_{m-i}^{-1}\frac{M_{m-i}}{(m+1)!}+\sum_{k=0}^{m-1-i}\beta_{m-1-k}^{-1}\begin{pmatrix} m-1-k \\ i\end{pmatrix}\frac{M_{m-1-i-k}}{(m-k)!}\right\}$$

$$=\sum_{l=0}^{m-i}\begin{pmatrix} i+l \\ i\end{pmatrix}\beta_l^{-1}\frac{M_l}{(i+l)!},\qquad 1\leqslant i\leqslant m-1.$$

In particular, consider the case that $\sigma(q)=\rho(q)\equiv 1,\ 0\leqslant q\leqslant 1$. Since

$$\bar{d}_{1,1}=-3M_2\left(\frac{M_1}{3!}+\frac{M_0}{2!}\right)+2M_1\frac{M_3}{4!}+\left(2\frac{M_1}{3!}+\frac{M_0}{2!}\right)M_2+\frac{M_0}{3!}M_3$$

$$=3M_2\left(\frac{M_3}{5!}+\frac{M_2}{4!}\right)+2M_1\frac{M_3}{4!}-\left(4\frac{M_3}{5!}+3\frac{M_2}{4!}\right)M_2+\frac{M_0}{3!}M_3$$

$$=\left(-\frac{M_2}{5!}+2\frac{M_1}{4!}+\frac{M_0}{3!}\right)M_3,$$

we have that by Corollary 2.1,

$$4\frac{M_3}{6!}+3\frac{M_2}{5!}+2\frac{M_1}{4!}+\frac{M_0}{3!}=\left(\frac{M_3}{5!}+\frac{M_2}{4!}+\frac{M_1}{3!}+\frac{M_0}{2!}\right)-2\left(\frac{M_3}{6!}+\frac{M_2}{5!}+\frac{M_1}{4!}+\frac{M_0}{3!}\right)=0,$$

$$\frac{M_3}{6!}+\frac{M_2}{5!}+C_1=\frac{1}{4}\left(4\frac{M_3}{6!}+3\frac{M_2}{5!}+2\frac{M_1}{4!}+\frac{M_0}{3!}\right)=0$$

and

$$\begin{aligned}\frac{M_3}{6!}+(C_2+C_3)&=\frac{1}{10}\left(10\frac{M_3}{6!}+6\frac{M_2}{5!}+3\frac{M_1}{4!}+\frac{M_0}{3!}\right)\\&=\frac{1}{10}\left\{-\left(\frac{M_3}{4!}+\frac{M_2}{3!}+\frac{M_1}{2!}+\frac{M_0}{1!}\right)+\frac{1}{2}\left(\frac{M_3}{5!}+\frac{M_2}{4!}+\frac{M_1}{3!}+\frac{M_0}{2!}\right)\right.\\&\quad\left.+\left(\frac{M_3}{6!}+\frac{M_2}{5!}+\frac{M_1}{4!}+\frac{M_0}{3!}\right)\right\}=0.\end{aligned}$$

Hence (2.19) is satisfied for $\{M_k\}_{k=0}^{m}$ corresponding to Gauss–Legendre polynomial of degree 3, that is,

$$M_3(t)=\frac{M_3}{3!}t^3+\frac{M_2}{2!}t^2+\frac{M_1}{1!}t+M_0=\frac{M_3}{20}P_3(2t-1),$$

which corresponds to Corollary 2.1.

2.3. Simplified conditions in special cases of $f(t)$

If in (1.1), $f(t)$ is a polynomial of t whose degree is equal to or less than m, then we generalize the results of Theorem 2.3 in [11] on the existence and to determine the collocation polynomial satisfying the condition $|v(h)-y(h)|=\mathrm{O}(h^{2m+1})$ (see also Theorem 2.3 in [10]). We can easily obtain the following lemma.

Lemma 2.5. *Assume that $f(t)$ is a polynomial of t whose degree r is equal to or less than m, and there is a constant δ_0 such that*

$$\begin{aligned}&\left(\prod_{j=0}^{m}\alpha_j\beta_j^{-1}\right)\delta_0+\left(\prod_{j=1}^{m}\alpha_j\beta_j^{-1}\right)\beta_0^{-1}f_0+\left(\prod_{j=2}^{m}\alpha_j\beta_j^{-1}\right)\beta_1^{-1}f_1\\&\quad+\cdots+\alpha_m\beta_m^{-1}\beta_{m-1}^{-1}f_{m-1}+\beta_m^{-1}f_m=0.\end{aligned}\tag{2.20}$$

Then, there is a polynomial $y_f(t)$ such that its degree is equal to the degree r of $f(t)$ and it satisfies

$$\begin{cases}y_f'(t)=ay_f(t)+\displaystyle\int_0^1 y_f(\sigma(q)t)\,\mathrm{d}\mu(q)+\int_0^1 y_f'(\rho(q)t)\,\mathrm{d}v(q)+f(t), & 0<t\leqslant h,\\ y_f(0)=\delta_0\end{cases}\tag{2.21}$$

and for $z(t) = y(t) - y_f(t)$, *we have that*

$$\begin{cases} z'(t) = az(t) + \int_0^1 z(\sigma(q)t)\,\mathrm{d}\mu(q) + \int_0^1 z'(\rho(q)t)\,\mathrm{d}\nu(q), \quad 0 < t \leqslant h, \\ z(0) = y(0) - \delta_0. \end{cases} \tag{2.22}$$

The collocation solutions $v(t)$ *and* $w(t)$ *to* (1.1) *and* (2.22) *are derived from the same collocation polynomial* $M_m(t)$ *and we have*

$$v(t) = w(t) + y_f(t), \quad 0 \leqslant t \leqslant h \tag{2.23}$$

and hence

$$v(t) - y(t) = w(t) - z(t), \quad 0 \leqslant t \leqslant h. \tag{2.24}$$

In particular, $\delta_0 = y(0)$ *in* (2.20) *if, and only if,* $\psi_{m+1} = 0$. *In this case, we have* $z(t) = w(t) \equiv 0$ *and* $y(t) = v(t) = y_f(t)$.

Proof. Let $y_f(t) = \sum_{n=0}^{m} (\delta_n/n!)t^n$, $f(t) = \sum_{n=0}^{m} (f_n/n!)t^n$ and

$$\begin{cases} \delta_1 &= \alpha_0\beta_0^{-1}\delta_0 + \beta_0^{-1}f_0, \\ \delta_2 &= \alpha_1\beta_1^{-1}\delta_1 + \beta_1^{-1}f_1, \\ & \vdots \\ \delta_m &= \alpha_{m-1}\beta_{m-1}^{-1}\delta_{m-1} + \beta_{m-1}^{-1}f_{m-1}. \end{cases}$$

Then by (2.20), $\alpha_m\beta_m^{-1}\delta_m + \beta_m^{-1}f_m = 0$, and $y_f(t)$ is a polynomial solution of (2.21) whose degree is equal to the degree of $f(t)$, and we get (2.22), (2.23) and (2.24). $\square$

Remark 2.3. Under the conditions in Lemma 2.5, $w(h)$ is the (m,m)-Padé approximant to $z(h)$ (see also Theorem 2.3 in [10]). If $f(t)$ is not a constant function, then $v(h) = w(h) + y_f(h)$ is not the (m,m)-Padé approximant to $y(h)$ but a $(2m,m)$-rational approximant to $y(h)$.

From Theorem 2.2 and Lemma 2.5, we can easily obtain the following useful theorem (see also [10]).

Theorem 2.3. *Assume that* $f(t)$ *is a polynomial of* t *whose degree* r *is equal to or less than* m. *If* $\psi_{m+1} = 0$, *then* $y(t) = y_f(t)$ *and* $v(t) = y_f(t)$ *satisfy* (2.5) *for any collocation polynomial* $M_m(t)$. *If* $\psi_{m+1} \not\equiv 0$, *then* $\{M_k\}_{k=0}^{m}$ *are determined by*

$$\sum_{k=0}^{m} \gamma_{k,n} = 0, \quad n = 1, 2, \ldots, m, \tag{2.25}$$

Table 1
The errors $e_i(h)$ for $m=2$ and $h=2^{-n}$

		$v(h)$		$\bar{Q}_{4,2}(h)$		collocation at Gauss points	
q	n	$e_1(h)$	$\frac{e_1(h)}{e_1(2h)}$	$e_2(h)$	$\frac{e_2(h)}{e_2(2h)}$	$e_3(h)$	$\frac{e_3(h)}{e_3(2h)}$
	1	5.005..E-06		3.616..E-05		5.005..E-06	
	2	1.877..E-07	0.0375..	1.236..E-06	0.0341..	1.877..E-07	0.0375..
1.0	3	6.434..E-09	0.0342..	4.045..E-08	0.0327..	6.434..E-09	0.0342..
	4	2.106..E-10	0.0327..	1.293..E-09	0.0319..	2.106..E-10	0.0327..
	5	6.7..E-12	0.0319..	4.09..E-11	0.0316..	6.7..E-12	0.0319..
	6	2.1..E-13	0.0316..	1.2..E-12	0.0314..	2.1..E-13	0.0316..
	1	5.498..E-06		9.153..E-06		1.258..E-05	
	2	2.108..E-07	0.0383..	3.205..E-07	0.0350..	7.496..E-07	0.0595..
0.9	3	7.313..E-09	0.0346..	1.063..E-08	0.0331..	4.453..E-08	0.0594..
	4	2.408..E-10	0.0329..	3.426..E-10	0.0322..	2.687..E-09	0.0603..
	5	7.7..E-12	0.0320..	1.08..E-11	0.0317..	1.646..E-10	0.0612..
	6	2.4..E-13	0.0316..	3.4..E-13	0.0314..	1.01..E-11	0.0618..
	1	7.573..E-06		8.683..E-07		2.940..E-05	
	2	2.962..E-07	0.0391..	1.362..E-08	0.0156..	1.976..E-06	0.0672..
0.8	3	1.037..E-08	0.0350..	1.311..E-10	0.00962..	1.272..E-07	0.0643..
	4	3.435..E-10	0.0331..	1.3..E-12	0.00992..	8.0507..E-09	0.0632..
	5	1.10..E-11	0.0321..	—	—	5.06..E-10	0.0628..
	6	3.5..E-13	0.0317..	—	—	3.17..E-11	0.0626..
	1	1.986..E-05		1.452..E-05		7.080..E-05	
	2	7.922..E-07	0.0398..	5.997..E-07	0.0412..	4.796..E-06	0.0677..
0.5	3	2.804..E-08	0.0353..	2.157..E-08	0.0359..	3.097..E-07	0.0645..
	4	9.334..E-10	0.0332..	7.234..E-10	0.0335..	1.963..E-08	0.0633..
	5	3.01..E-11	0.0322..	2.34..E-11	0.0323..	1.234..E-09	0.0628..
	6	9.5..E-13	0.0317..	7.4..E-13	0.0318..	7.73..E-11	0.0626..
	1	2.719..E-05		2.699..E-05		4.780..E-05	
	2	1.080..E-06	0.0397..	1.073..E-06	0.0397..	2.710..E-06	0.0566..
0.2	3	3.817..E-08	0.0353..	3.794..E-08	0.0353..	1.531..E-07	0.0565..
	4	1.269..E-09	0.0332..	1.2620..E-09	0.0332..	8.915..E-09	0.0581..
	5	4.09..E-11	0.0322..	4.069..E-11	0.0322..	5.339..E-10	0.0598..
	6	1.2..E-12	0.0317..	1.2..E-12	0.0317..	3.25..E-11	0.0610..

where

$$\gamma_{k,n} = \beta_{k+n-1}^{-1}\left(\prod_{j=1}^{n-1}\alpha_{k+j}\beta_{k+j-1}^{-1}\right)\frac{M_k}{(k+n)!}, \quad k=0,1,2,\ldots,m, \;\; n=1,2,\ldots,m.$$

In this case, for (2.23), $w(h)$ *is the* (m,m)*-Padé approximant* $P_{m,m}(h)$ *determined by Theorem* 2.1 *to* $z(h)$ *in Lemma* 2.5, *and* $v(h)=P_{m,m}(h)+y_f(h)$, *and* $|v(h)-y(h)|=\mathrm{O}(h^{2m+1})$.

Remark 2.4. If $\psi_{m+1} \neq 0$, and $\{M_k\}_{k=0}^m$ are solutions of the equation $\sum_{k=0}^m \gamma_{k,n} = 0$, $n = 1, 2, \ldots, m$, then $M_{m,n}(t) = \sum_{k=0}^m \gamma_{k,n} t^{k+n}$, $n = 1, 2, \ldots, m$ satisfy the following equations:

$$\begin{cases} M_{m,1}(t) - \displaystyle\int_0^t \int_0^1 M'_{m,1}(\rho(q)\tau)\,\mathrm{d}\nu(q)\,\mathrm{d}\tau = \int_0^t M_m(x)\,\mathrm{d}x \\ M_{m,n}(t) - \displaystyle\int_0^t \int_0^1 M'_{m,n}(\rho(q)\tau)\,\mathrm{d}\nu(q)\,\mathrm{d}\tau = \int_0^t \left\{ aM_{m,n-1}(x) + \int_0^1 M_{m,n-1}(\sigma(q)x)\,\mathrm{d}\mu(q) \right\} \mathrm{d}x, \\ \quad n = 2, 3, \ldots, m, \end{cases}$$

and Eq. (2.14) is expressed as $M_{m,n}(1) = 0$, $n = 1, 2, \ldots, m$. Hence Theorems 2.2 and 2.3 are some extensions of Theorem 3.3 in Brunner [4], Theorem 2.3 in Takama et al. [11] and Ishiwata [10].

3. Numerical experiments

Now we show a numerical example (cf. [5]).

Example 3.1. Consider the following PIDE:

$$y'(t) = -y(t) + \frac{q}{2} y(qt) - \frac{q}{2} \mathrm{e}^{-qt}, \quad y(0) = 1,$$

where $y(t) = \mathrm{e}^{-t}$. For $q = 1.0, 0.9, 0.8, 0.5, 0.2$, we test our Theorem 2.2 by using the collocation polynomial determined by (2.15). For $m = 2$ and $h = 2^{-n}$, $n = 1, 2, 3, 4, 5, 6$, the errors $e_1(h) = v(h) - y(h)$, $e_2(h) = \bar{Q}_{4,2}(h) - y(h)$ are shown in Table 1. Note that $1/2^5 = 0.03125$ and $1/2^4 = 0.0625$. Hence we can see that both errors $e_1(h)$ and $e_2(h)$ imply $\mathrm{O}(h^5)$ as denoted in Theorem 2.2. On the other hand, for the collocation at Gauss points, we see that the error $e_3(h)$ implies $\mathrm{O}(h^4)$ for $0 < q < 1$.

Remark 3.1. Note that for the case $q = 0.8$, $\bar{Q}_{3,2}(h) \doteq R_{3,2}(h)$, and hence the error implies $\mathrm{O}(h^6)$.

These results for the first mesh point $t = h$, can be applied to the computation and global error analysis of the successive mesh points. We will consider these in future work (see [2] for a "quasi-geometric meshes" and cf. [5] for a "geometric meshes").

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